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# An identity on SU(2) invariants

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## Abstract

We prove an identity (equation (1) below) among SU(2)  $6j$  and  $9j$  symbols that generalizes the Biedenharn–Elliott sum rule. We prove the result using diagrammatic techniques (briefly reviewed here), and then provide an algebraic proof. This identity is useful for studying meson–baryon scattering in which an extra isoscalar meson is produced.

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## 1. Introduction

In this paper we prove the following identity for a particular sum over two  $9j$  and one  $6j$  symbol which, to the best of our knowledge, seems not to appear previously as a distinct entity in the literature:

$$\begin{aligned} & \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} \begin{Bmatrix} j_3 & j_4 & j_5 \\ \Delta & j'_5 & j'_4 \end{Bmatrix} \begin{Bmatrix} j'_1 & j'_2 & j_3 \\ j'_4 & j'_5 & j'_6 \end{Bmatrix} \\ &= \sum_{k,\ell} [k][\ell] (-1)^\Phi \begin{Bmatrix} j_1 & j'_1 & k \\ j_6 & j'_6 & \ell \\ j_5 & j'_5 & \Delta \end{Bmatrix} \begin{Bmatrix} j_2 & j'_2 & k \\ j_6 & j'_6 & \ell \\ j_4 & j'_4 & \Delta \end{Bmatrix} \begin{Bmatrix} j_1 & j_2 & j_3 \\ j'_2 & j'_1 & k \end{Bmatrix}, \end{aligned} \quad (1)$$

where  $[j] \equiv 2j+1$  is the multiplicity of the spin- $j$  irreducible representation, and the argument of the phase  $(-1)^\Phi$  is given by

$$\Phi = j_1 + j_2 + j_6 - j'_4 - j'_5 - j'_6 + k - \Delta. \quad (2)$$

This identity is a generalization of the well-known Biedenharn–Elliott (BE) sum rule (equation (8) below), to which (as we shall show) it reduces when  $\Delta$  is set to zero.

This identity arises when considering the angular momentum coupling scheme:

$$\begin{aligned} \mathbf{j}_3 &= \mathbf{j}_1 + \mathbf{j}_2, & \mathbf{j}_3 &= \mathbf{j}'_1 + \mathbf{j}'_2, & \mathbf{j}'_1 &= \mathbf{j}_1 + \mathbf{k}, \\ \mathbf{j}_6 &= \mathbf{j}_2 + \mathbf{j}_4, & \mathbf{j}'_6 &= \mathbf{j}'_2 + \mathbf{j}'_4, & \mathbf{j}_2 &= \mathbf{j}'_2 + \mathbf{k}, \\ \mathbf{j}_5 &= \mathbf{j}_3 + \mathbf{j}_4, & \mathbf{j}'_5 &= \mathbf{j}_3 + \mathbf{j}'_4, & \mathbf{j}'_4 &= \mathbf{j}_4 + \Delta \\ \mathbf{j}_5 &= \mathbf{j}_1 + \mathbf{j}_6, & \mathbf{j}'_5 &= \mathbf{j}'_1 + \mathbf{j}'_6, & \mathbf{j}'_5 &= \mathbf{j}_5 + \Delta, \\ \mathbf{j}'_6 &= \mathbf{j}_6 + \ell, & \Delta &= \mathbf{k} + \ell. \end{aligned} \quad (3)$$

The unprimes and primes suggest (for example) initial- and final-state quantum numbers respectively, whose inequality is enforced by a ‘spurion’  $\Delta$ . Other physically useful coupling schemes may be obtained by globally replacing any of these angular momenta  $\mathbf{j}$  by their time-reversed forms  $\tilde{\mathbf{j}} \equiv -\mathbf{j}$ , where  $\tilde{\mathbf{j}}$  is the angular momentum operator whose eigenstates are related to those ( $|jm\rangle$ ) of  $\mathbf{j}$  by  $(-1)^{j+m} |j-m\rangle$ . This manipulation gives a well-defined meaning to the concept of subtracting angular momentum operators [1].

As an explicit physical example using this coupling scheme, consider meson-baryon scattering ( $\phi B \rightarrow \phi' B'$ ) in which an additional isoscalar meson  $f$  is produced with total angular momentum  $J_f$  with respect to the other final-state particles:  $\phi B \rightarrow \phi' B' f$ . The specified observables are isospins and angular momenta:  $I_{\phi(\phi')}, J_{\phi(\phi')}$  for the mesons (as usual,  $J$  denotes spin and orbital angular momenta combined) and  $I_{B(B')}, S_{B(B')}$  for the baryons. The total  $s$ -channel quantum numbers are  $\mathbf{I}_s \equiv \mathbf{I}_\phi + \mathbf{I}_B = \mathbf{I}_{\phi'} + \mathbf{I}_{B'}, \mathbf{J}_s = \mathbf{J}_\phi + \mathbf{S}_B, \mathbf{J}'_s = \mathbf{J}_{\phi'} + \mathbf{S}_{B'}$  and  $\mathbf{J}'_s = \mathbf{J}_s - \mathbf{J}_f$ . In a chiral soliton model or the  $1/N_c$  expansion of QCD, the vector sum of isospin and angular momentum for each particle assumes extra significance: the stable baryons (such as nucleons) are zero-eigenvalue states of the operators  $\mathbf{I}_B + \mathbf{S}_B$  and  $\mathbf{I}_{B'} + \mathbf{S}_{B'}$ , and the scattering is characterized by the ‘grand spins’  $\mathbf{K} \equiv \mathbf{I}_s + \mathbf{J}_s, \mathbf{K}' \equiv \mathbf{I}_s + \mathbf{J}'_s$ . The application of equation (1) arises when one considers processes such as this not in the  $s$ -channel but the  $t$ -channel [2]: then  $\mathbf{I}_{\phi'} = \mathbf{I}_\phi + \mathbf{I}_t$  and  $\mathbf{J}_{\phi'} = \mathbf{J}_\phi + \mathbf{J}_t$ . The full identification using the notation of equation (3) is  $\mathbf{j}_1 \rightarrow \mathbf{I}_\phi, \mathbf{j}_2 \rightarrow \mathbf{I}_B, \mathbf{j}_3 \rightarrow \mathbf{I}_s, \mathbf{j}_4 \rightarrow \mathbf{J}_s, \mathbf{j}_5 \rightarrow \mathbf{K}, \mathbf{j}_6 \rightarrow \mathbf{J}_\phi$  (and analogously for the primed  $\mathbf{j}$ 's), and  $\mathbf{k} \rightarrow \mathbf{I}_t, \ell \rightarrow \mathbf{J}_t$ , and  $\Delta \rightarrow -\mathbf{J}_f$ . The transcription of equation (1) then reads

$$\begin{aligned} & \begin{Bmatrix} I_\phi & I_B & I_s \\ J_s & K & J_\phi \end{Bmatrix} \begin{Bmatrix} I_s & J_s & K \\ J_f & K' & J'_s \end{Bmatrix} \begin{Bmatrix} I_{\phi'} & I_{B'} & I_s \\ J'_s & K' & J_{\phi'} \end{Bmatrix} \\ &= \sum_{I_t, J_t} [I_t][J_t] (-1)^\Phi \begin{Bmatrix} I_\phi & I_{\phi'} & I_t \\ J_\phi & J_{\phi'} & J_t \\ K & K' & J_f \end{Bmatrix} \begin{Bmatrix} I_B & I_{B'} & I_t \\ J_\phi & J_{\phi'} & J_t \\ J_s & J'_s & J_f \end{Bmatrix} \begin{Bmatrix} I_\phi & I_B & I_s \\ I_{B'} & I_{\phi'} & I_t \end{Bmatrix}, \quad (4) \end{aligned}$$

where now  $\Phi = I_\phi + I_B + J_\phi - J'_s - K' - J_{\phi'} + I_t - J_f$ . The three  $6j$  symbols on the left-hand side (expressed solely in terms of  $s$ -channel quantities) appear as coefficients in expressions for partial-wave scattering amplitudes for the process  $\phi B \rightarrow \phi' B' f$  written in terms of underlying ‘reduced’ amplitudes labeled by  $K$  values; for example, the case in which  $f$  is absent (effectively,  $J_f = 0$ ) has been studied for quite some time [3]. Expressing amplitudes in terms of  $t$ -channel quantities (as is manifest on the right-hand side of equation (4)) is of particular interest because such amplitudes scale as  $1/N_c^{|I_t - J_t|}$  [2], thereby creating a hierarchy of dominant and subdominant amplitudes in the  $1/N_c$  expansion of QCD.

The most illuminating proof of equation (1) uses diagrammatic techniques, an approach we summarize in section 2. We present the diagrammatic proof in section 3, and finally an algebraic proof, using standard SU(2) identities, in section 4.

## 2. Diagrammatic method for coupling angular momenta

### 2.1. Notation

Algebraic techniques for manipulating Clebsch–Gordan coefficients (CGC) to obtain invariants (quantities independent of magnetic quantum numbers  $m$ , such as  $6j$  and  $9j$  symbols) are certainly straightforward and appear in all standard treatments of the topic [1]. However, at a certain point of complexity these techniques become particularly cumbersome, and the bookkeeping necessary to impose the required identities for simplifying such expressions (particularly with regard to the numerous phases  $(-1)^n$  that arise) becomes increasingly

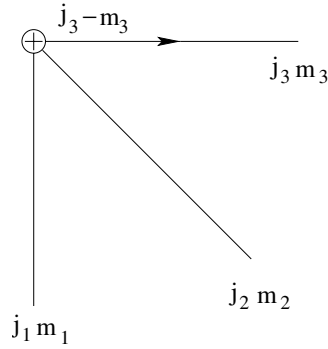


Figure 1. Graphical representation of the  $3j$  symbol.

onerous. A much cleaner strategy is to use diagrammatic techniques introduced originally by Jucys (alternate spellings *Yutsis*, *Iutsis*), Levinson and Vanagas (JLV) [4]. In this method, each angular momentum  $j$  is represented as a line, and each vertex represents a  $3j$  symbol (or CGC). The quantum number  $m$  corresponding to  $j$  is summed if and only if line  $j$  is internal to the diagram.

The diagrammatic technique is particularly valuable because of two features: first, the identities involved in combining large complexes of angular momenta become topological in nature, and the ability to identify them reduces to one’s cunning in picturing how to connect the lines. Second, the phases endemic to CGC are incorporated in the diagrams very neatly (as too are factors of  $[j]$ , but we do not need them here), appearing as either signs at the vertices or arrows on the lines, in the manner described below. The JLV technique is laid out pedagogically in the text by Lindgren and Morrison (LM) [5] or the review by Wormer and Paldus [6]. Here we list only the features essential for this paper.

For starters, the vertex in figure 1 represents a  $3j$  symbol:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{pmatrix} (-1)^{j_3 - m_3}, \tag{5}$$

with the specific ordering of  $j$ ’s and  $m$ ’s as (counter)clockwise defining the vertex orientation as (positive) negative, as indicated by a sign at the vertex in the diagram. The arrow on  $j_3$  introduces the phase  $(-1)^{j_3 - m_3}$ ; were it pointing toward the vertex, the phase introduced would be  $(-1)^{j_3 + m_3}$ . Note that we follow the LM arrow convention, which is opposite that of JLV convention, since as argued in [5] it is more closely analogous to the flow of momentum in scattering diagrams.

Several manipulations help us simplify such calculations:

- (1) Two arrows pointing in opposite direction on the same line can be removed.
- (2) Reversing the direction of an arrow introduces an additional  $(-1)^{2j}$ .
- (3) Reversing the orientation of the vertex (changing the sign symbol) introduces an additional  $(-1)^{j_1 + j_2 + j_3}$ .
- (4) Introducing three arrows all pointing inward or outward at a vertex does not change the value of the diagram.

## 2.2. Invariants

The combination of several vertices (with no external lines) forms a diagram representing a higher-order  $3nj$  symbol; for example, the irreducible combination of four vertices, as depicted

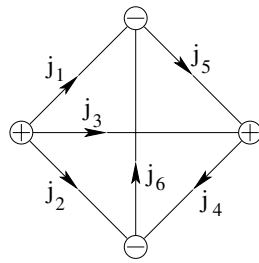


Figure 2. Graphical representation of the  $6j$  symbol.

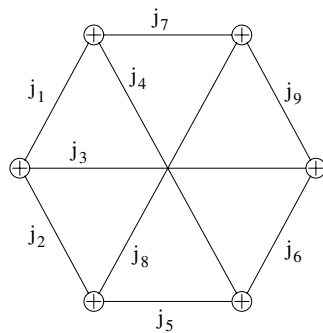


Figure 3. Graphical representation of the  $9j$  symbol.

in figure 2, forms the  $6j$  symbol,

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}, \tag{6}$$

and the irreducible combination of six vertices, as depicted in figure 3, forms the  $9j$  symbol,

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \\ j_7 & j_8 & j_9 \end{Bmatrix}. \tag{7}$$

### 2.3. Theorems

The true power of the JLV approach arises through a series of theorems [4, 5], reminiscent of the factorization theorems of quantum field theory, that allow one to cut diagrams internally connected only by a small number  $n$  of lines. Consider a diagram consisting of two such blocks,  $\bar{\alpha}$  and  $\beta$ , such that  $\bar{\alpha}$  is *closed* (no external lines) and in *normal form* (every internal line on  $\bar{\alpha}$  carries an arrow, and any arrows on the lines connecting with  $\beta$  are pushed into the block  $\beta$ ). Then one obtains a series of theorems  $JLV_n$ ,  $n = 1, 2, \dots$ . Of greatest interest to us here are  $JLV_3$  (depicted in figure 4) and  $JLV_4$  (figure 5).  $JLV_3$ , for example, applied to a system in which block  $\beta$  is empty, turns out to be none other than the Wigner–Eckart theorem.

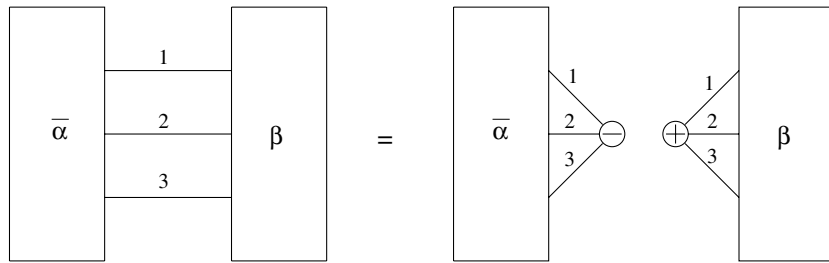


Figure 4. The JLV3 theorem.

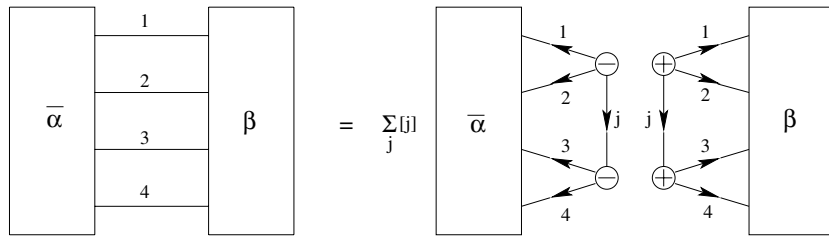


Figure 5. The JLV4 theorem.

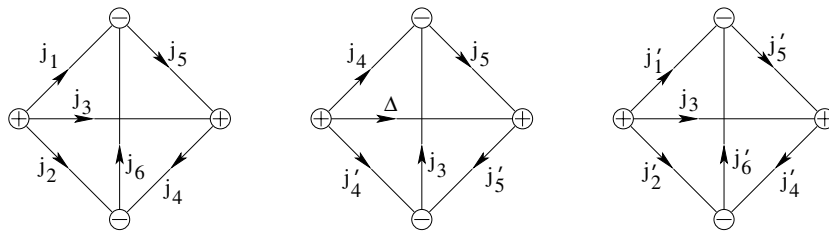


Figure 6. The three  $6j$  symbols of equation (9).

### 3. Proof of the identity

The Biedenharn–Elliott sum rule [1] reads

$$\begin{aligned} & \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} \begin{Bmatrix} j'_1 & j'_2 & j_3 \\ j_4 & j_5 & j'_6 \end{Bmatrix} \\ &= \sum_{\mathcal{J}} (-1)^{\sigma[\mathcal{J}]} \begin{Bmatrix} j_1 & j_6 & j_5 \\ j'_6 & j'_1 & \mathcal{J} \end{Bmatrix} \begin{Bmatrix} j_2 & j_6 & j_4 \\ j'_6 & j'_2 & \mathcal{J} \end{Bmatrix} \begin{Bmatrix} j_1 & j_2 & j_3 \\ j'_2 & j'_1 & \mathcal{J} \end{Bmatrix}, \end{aligned} \tag{8}$$

where  $\sigma$  is the sum of the nine distinct arguments on the left-hand side, plus  $\mathcal{J}$ . Here we wish to find the analog of the BE sum rule for the following three  $6j$  symbols, the left-hand side of equation (1), represented graphically in figure 6:

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} \begin{Bmatrix} j_3 & j_4 & j_5 \\ \Delta & j'_5 & j'_4 \end{Bmatrix} \begin{Bmatrix} j'_1 & j'_2 & j_3 \\ j'_4 & j'_5 & j'_6 \end{Bmatrix}. \tag{9}$$

By means of the JLV3 theorem, the three  $6j$  symbols of equation (9) combine into a diagram with 12 quantum numbers shown in figure 7. In order to introduce the quantum number  $k$  of equation (1), we cut the diagram on the four lines,  $j_1, j'_1, j_2$  and  $j'_2$  using the

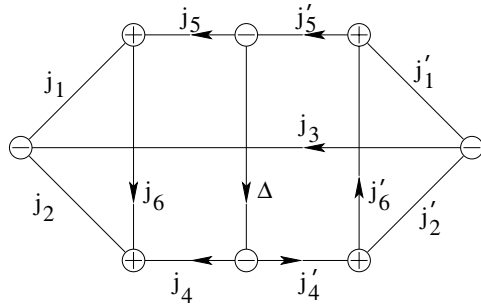


Figure 7. Result of combining the three  $6j$  symbols of equation (9) or figure 6.

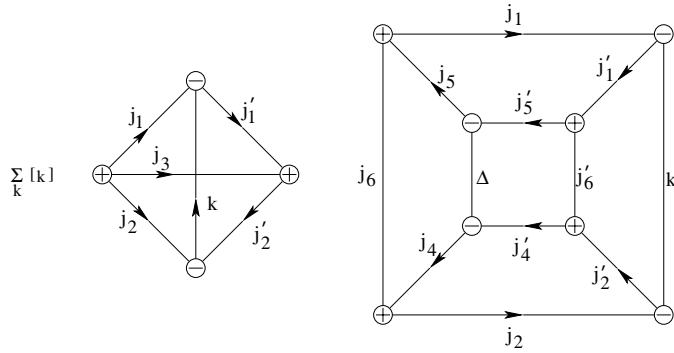


Figure 8. The result of applying JLV4 to figure 7.

JLV4 theorem. The summed quantum number in this diagram is indeed the desired  $k$ , and the result is a  $6j$  symbol and a  $12j$  symbol of the second kind, as shown in figure 8. Since the latter object is surely obscure to most readers, we pause to point out that, while higher  $3nj$  symbols are not difficult to generate and manipulate using the diagrammatic approach, and while they possess remarkable and intricate symmetry properties, they may always be reduced to products over convenient sums of  $6j$  and  $9j$  symbols using the JLV theorems [7, 8]. One other fine point is that the attractive square diagram in figure 8 actually differs from the true  $12j$  symbol by a phase  $(-1)^{j_1-j_2+j'_4-j'_5}$ , but this distinction is purely formal; like equation (1), the diagram as depicted is symmetric upon exchange of primed and unprimed quantum numbers.

To introduce the quantum number  $\ell$ , we make another four-line cut, here on  $j_6, j'_6, k$  and  $\Delta$  in the  $12j$  symbol of figure 8. The result of this action is shown in figure 9. The hexagonal figures are none other than standard  $9j$  symbols in canonical JLV form.

In fact, we have expressed a particular product of three  $6j$  symbols (equation (9)) as a  $6j$  and two  $9j$  symbols summed over two new quantum numbers,  $k$  and  $\ell$ . To repeat equation (1),

$$\begin{aligned} & \left\{ \begin{matrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{matrix} \right\} \left\{ \begin{matrix} j_3 & j_4 & j_5 \\ \Delta & j'_5 & j'_4 \end{matrix} \right\} \left\{ \begin{matrix} j'_1 & j'_2 & j_3 \\ j'_4 & j'_5 & j'_6 \end{matrix} \right\} \\ &= \sum_{k,\ell} [k][\ell] (-1)^\Phi \left\{ \begin{matrix} j_1 & j'_1 & k \\ j_6 & j'_6 & \ell \\ j_5 & j'_5 & \Delta \end{matrix} \right\} \left\{ \begin{matrix} j_2 & j'_2 & k \\ j_6 & j'_6 & \ell \\ j_4 & j'_4 & \Delta \end{matrix} \right\} \left\{ \begin{matrix} j_1 & j_2 & j_3 \\ j'_2 & j'_1 & k \end{matrix} \right\}, \end{aligned} \tag{10}$$

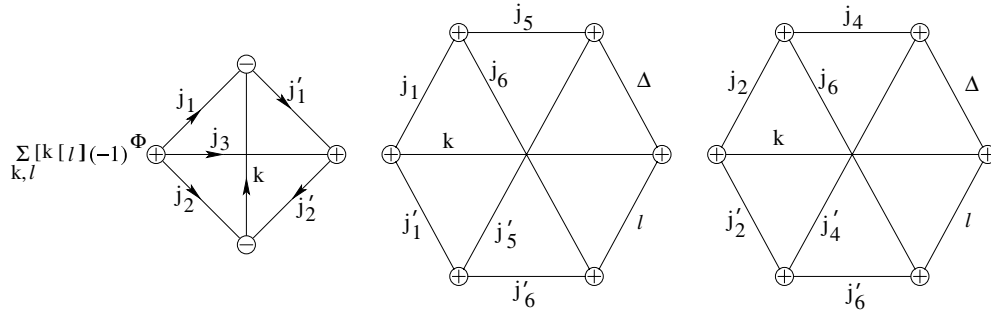


Figure 9. The result of applying JLV4 to figure 8, which is expressed algebraically in equation (1).

with the phase given by

$$\Phi = j_1 + j_2 + j_6 - j'_4 - j'_5 - j'_6 + k - \Delta. \tag{11}$$

While  $\Phi$  is not primed–unprimed symmetric, neither are the two new  $9j$  symbols, because switching their initial and final quantum numbers requires column exchanges; when the necessary permutations are taken into account, it is quite straightforward to show this explicit symmetry. To the best of our knowledge, equation (1) actually represents a new  $SU(2)$  identity<sup>1</sup>, one that reduces for  $\Delta = 0$  to the BE sum rule. This reduction becomes apparent when one simplifies using special cases

$$\begin{Bmatrix} j_3 & j_4 & j_5 \\ 0 & j'_5 & j'_4 \end{Bmatrix} = \frac{(-1)^{j_3+j_4+j_5}}{\sqrt{[j_4][j_5]}} \delta_{j_4 j'_4} \delta_{j_5 j'_5}, \tag{12}$$

$$\begin{Bmatrix} j_1 & j'_1 & k \\ j_6 & j'_6 & \ell \\ j_5 & j'_5 & 0 \end{Bmatrix} = \frac{(-1)^{j'_1+j_6+j_5+k}}{\sqrt{[k][j_5]}} \delta_{j_5 j'_5} \delta_{k \ell} \begin{Bmatrix} j_1 & j_6 & j_5 \\ j'_6 & j'_1 & k \end{Bmatrix}, \tag{13}$$

and

$$\begin{Bmatrix} j_2 & j'_2 & k \\ j_6 & j'_6 & \ell \\ j_4 & j'_4 & 0 \end{Bmatrix} = \frac{(-1)^{j'_2+j_6+j_4+k}}{\sqrt{[k][j_4]}} \delta_{j_4 j'_4} \delta_{k \ell} \begin{Bmatrix} j_2 & j_6 & j_4 \\ j'_6 & j'_2 & k \end{Bmatrix}, \tag{14}$$

in which case both  $k$  and  $\ell$  reduce to  $\mathcal{J}$  of equation (8).

#### 4. Algebraic proof

Equation (1) is also fairly straightforward to verify algebraically, once the right-hand side is known. Here we reduce this side of the equation. We use the symmetry properties of  $6j$  symbols and  $9j$  symbols:  $6j$  symbols are invariant under the permutation of any two columns or under the exchange of upper and lower entries for any two columns, while  $9j$  symbols are invariant under even permutations of any two rows or columns. We also use the BE sum rule, equation (8). Furthermore,  $9j$  symbols may be expanded in terms of  $6j$  symbols using the standard identity [1]

$$\begin{Bmatrix} j_6 & j'_6 & \ell \\ j_4 & j'_4 & \Delta \\ j_2 & j'_2 & k \end{Bmatrix} = \sum_x (-1)^{2x} [x] \begin{Bmatrix} j_6 & j_4 & j_2 \\ j'_2 & k & x \end{Bmatrix} \begin{Bmatrix} j'_6 & j'_4 & j'_2 \\ j_4 & x & \Delta \end{Bmatrix} \begin{Bmatrix} \ell & \Delta & k \\ x & j_6 & j'_6 \end{Bmatrix}, \tag{15}$$

<sup>1</sup> Similar identities appear in the literature, but equation (1) appears to be distinct.



where the  $9j$  symbol is equivalent to the second one in equation (1). The first  $9j$  symbol of equation (1) and the last  $6j$  symbol of equation (15) (arguments rearranged), along with their sum over  $\ell$ , may be re-expressed using another standard identity [1]:

$$\sum_{\ell} [\ell] \begin{Bmatrix} j_6 & j_6' & \ell \\ j_5 & j_5' & \Delta \\ j_1 & j_1' & k \end{Bmatrix} \begin{Bmatrix} j_6 & j_6' & \ell \\ \Delta & k & x \end{Bmatrix} = (-1)^{2x} \begin{Bmatrix} j_5 & j_5' & \Delta \\ j_6' & x & j_1' \end{Bmatrix} \begin{Bmatrix} j_1 & j_1' & k \\ x & j_6 & j_5 \end{Bmatrix}. \quad (16)$$

The phase  $(-1)^{2x}$  cancels between equations (15) and (16), and the remaining expression reads

$$\sum_x [x] \sum_k [k] (-1)^{\Phi} \begin{Bmatrix} j_6' & j_4 & j_2' \\ j_4 & x & \Delta \end{Bmatrix} \begin{Bmatrix} j_5 & j_5' & \Delta \\ j_6' & x & j_1' \end{Bmatrix} \begin{Bmatrix} j_6 & j_4 & j_2 \\ j_2 & k & x \end{Bmatrix} \begin{Bmatrix} j_1 & j_1' & k \\ x & j_6 & j_5 \end{Bmatrix} \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_2' & j_1' & k \end{Bmatrix}. \quad (17)$$

The summed angular momentum  $k$  appears only in the last three of these  $6j$  symbols, and the phase argument  $\Phi$  (equation (2)) conveniently contains a factor of  $k$ , suggesting that they can be simplified using the BE sum rule. Indeed, writing the sum of the ten angular momenta in the last three  $6j$  symbols as  $\sigma$ , one has

$$\sum_k (-1)^{\sigma} [k] \begin{Bmatrix} j_2 & j_6 & j_4 \\ x & j_2' & k \end{Bmatrix} \begin{Bmatrix} j_6 & j_5 & j_1 \\ j_1' & k & x \end{Bmatrix} \begin{Bmatrix} j_2 & j_1 & j_3 \\ j_1' & j_2' & k \end{Bmatrix} = \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} \begin{Bmatrix} j_4 & j_5 & j_3 \\ j_1' & j_2' & x \end{Bmatrix}. \quad (18)$$

The first  $6j$  symbol appears on the left-hand side of equation (1). The remaining factors to be simplified (those containing  $x$ ) are the first two  $6j$  symbols in equation (17), the second in equation (18) and the phase  $(-1)^{\Phi-\sigma} = (-1)^{\sigma-\Phi}$  (since  $\Phi$  and  $\sigma$  are integers)  $\equiv (-1)^{\tilde{\sigma}}$ . The BE sum rule again simplifies the expression:

$$\sum_x (-1)^{\tilde{\sigma}} [x] \begin{Bmatrix} j_4 & \Delta & j_4' \\ j_6' & j_2' & x \end{Bmatrix} \begin{Bmatrix} \Delta & j_5' & j_5 \\ j_1' & x & j_6' \end{Bmatrix} \begin{Bmatrix} j_4 & j_5 & j_3 \\ j_1' & j_2' & x \end{Bmatrix} = \begin{Bmatrix} j_3 & j_4 & j_5 \\ \Delta & j_5' & j_4' \end{Bmatrix} \begin{Bmatrix} j_1' & j_2' & j_3 \\ j_4' & j_5' & j_6' \end{Bmatrix}, \quad (19)$$

since  $\tilde{\sigma}$  is easily shown to be the sum of the ten angular momenta on the left-hand side. These two  $6j$  symbols, along with the first from equation (18), complete the left-hand side of equation (1), and hence complete the proof.

### Acknowledgment

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